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Approximation by Linear Combinations of Continuous Functions with Restricted Coefficients

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We prove necessary and sufficient conditions for linear combinations of given functions $K_k \in C[a, b]$ $(k = 1, 2, ...)$ to be dense in $C[a, b]$, $C_0[a, b]$ respectively, where the coefficients satisfy bound constraints. Our general approach to such problems is based on a new method of functional analysis concerning a relationship between approximations in normed linear spaces with generalized restrictions, defined by seminorms, and certain corresponding properties of bounded linear functionals on these spaces. In the special case of approximation by Müntz polynomials with restricted coefficients some known results and sharpened versions of these can be deduced from our general theorems. Finally, an application to another type of function K_k is obtained, where $K_k(t) = \phi(t\lambda_k^{-1})$, $\lambda_k \to \infty$, with ϕ being certain analytic functions.

Let $C[a, b]$, $-\infty < a < \infty$, denote the set of all complex-valued continuous functions on $[a, b]$, and let $C_0[a, b]$ be the set of all $f \in C[a, b]$ with $f(a) = 0$.

If $K = (K_k)$ is a sequence of functions $K_k \in C[a, b]$, and $D = (D_k)$ is a sequence of numbers $D_k > 0$ $(k = 1, 2, ...)$, we define $P_{K,D}$ to be the class of all linear combinations g, $g(t) = \sum_{k=1}^{N} a_k K_k(t)$ (a_k complex), with the restriction that $|a_k| \leqslant D_k$ $(k = 1, 2, ..., N)$.

We ask for necessary and sufficient conditions on $K = (K_k)$ and $D = (D_k)$ for $P_{K,D}$ to be dense in $C_0[a, b]$ or in $C[a, b]$ in the uniform norm.

This problem has been completely solved at first in the special case $K_k(t) = t^k$, $[a, b] = [0, 1]$, and the following result was proved in [2, Theorem 1] and [7] by using Bernstein polynomials.

THEOREM 1. *Suppose that $K_k(t) = t^k$ and $D_k = A_k^k$, $A_k > 0$ $(k = 1, 2, ...)$. Then $P_{K,D}$ is dense in $C_0[0, 1]$ if and only if there exists a subsequence (k_i) of (k) such that*

$$\sum_{i=1}^{\infty} k_i^{-1} = \infty \qquad \text{and} \qquad A_{k_i} \to \infty \ (i \to \infty).$$

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An analogous problem for the approximation by polynomials of a monotonically decreasing sequence was solved in [1].

In this paper we give a general approach by extending a method of functional analysis, which was introduced in [3, 4]. Our general method is based on a relationship between approximations with generalized restrictions, defined by seminorms, and certain corresponding properties of bounded linear functionals on normed linear spaces.

We shall obtain the following five results as applications of our general theorems:

THEOREM 2. *Suppose that* $K=(K_k)$, $K_k \in C[a,b]$, $-\infty<a<b<\infty$, *and* $D_k>0$ $(k=1,2,\ldots)$. *Then* $P_{K,D}$ *is dense in* $C_0[a,b]$ *if and only if the following condition* (B) *is satisfied*:

$$\text{(B)} \qquad \sum_{k=1}^{\infty} D_k \left| \int_a^b K_k(t)\, d\alpha(t) \right| < \infty, \qquad \int_a^b |d\alpha(t)| < \infty$$

> *always implies* $\alpha(t)=0$ *on* $(a,b]$ *for a normalized* α *on* $[a,b]$, *where* α *is normalized if* $\alpha(t)=2^{-1}[\alpha(t+0)+\alpha(t-0)]$ *for* $t\in(a,b)$ *with* $\alpha(b)=0$.

Moreover, $P_{K,D}$ *is dense in* $C[a,b]$ *if and only if the inequalities of* (B) *imply* $\alpha(t)=0$ *on* $[a,b]$ *for each normalized* α, *which is satisfied if and only if* (B) *is valid and* $\sum_{k=1}^{\infty} D_k\,|K_k(a)|=\infty$.

It follows easily from Theorem 2 that $\sum_{k=1}^{\infty} D_k\,|K_k(t)|=\infty$ for all $t\in(a,b]$ or all $t\in[a,b]$ is a necessary condition for $P_{K,D}$ to be dense in $C_0[a,b]$, $C[a,b]$ respectively. This is obvious if we choose for each fixed t the function α such that $\alpha(u)=0$ $(u>t)$ and $\alpha(u)=-1$ $(u<t)$.

We deduce from Theorem 2 four special results by using properties of linear functionals on $C[a,b]$. First, we give a new proof for the following theorem, which is contained in a more general result concerning complex exponents due to [2, Theorem 3].

THEOREM 3. *If* $K_k(t)=t^{\lambda_k}$, $\lambda_k>0$, $\lambda_{k+1}-\lambda_k\geqslant c>0$, $\sum_{k=1}^{\infty}\lambda_k^{-1}=\infty$, *and* $D_k=A_k^{\lambda_k}$ *with* $A_k>0$ $(k=1,2,\ldots)$, *then* $P_{K,D}$ *is dense in* $C_0[0,1]$ *if and only if there exists a subsequence* (k_i) *of* (k) *such that*

$$\sum_{i=1}^{\infty} \lambda_{k_i}^{-1}=\infty \text{ and } A_{k_i}\to\infty \ (i\to\infty). \qquad (1)$$

We can improve [2, Theorem 2, Theorem 4 in the real case] and results of [4] by

THEOREM 4. *Suppose that* $K_k(t)=t^{\lambda_k}$, $\lambda_k>0$, $\lambda_{k+1}-\lambda_k\geqslant c>0$, $\sum_{k=1}^{\infty}\lambda_k^{-1}=\infty$, *and* $D_k=c_k a^{-\lambda_k}$ *with* $a>0$, $c_k>0$ $(k=1,2,\ldots)$. *Then* $P_{K,D}$ *is dense in* $C_0[a,b]$, $0<a<b$, *if and only if there exists a subsequence* (k_i) *of* (k) *such that*

$$\sum_{i=1}^{\infty}\lambda_{k_i}^{-1}=\infty \qquad \text{and} \qquad \varliminf_{i\to\infty} c_{k_i}^{1/\lambda_{k_i}}\geqslant 1. \tag{2}$$

In order that $P_{K,D}$ *is dense in* $C[a,b]$ *it is necessary and sufficient that there exists a subsequence* (k_i) *satisfying* (2), *and that* $\sum_{k=1}^{\infty}c_k=\infty$.

Concerning Theorem 4 it is interesting that $P_{K,D}$ can be dense in $C_0[a,b]$ even if $c_k\to 0$ $(k\to\infty)$, for example in the case $c_k=k^{-\beta}$, $\beta>0$.
For bounded λ_k we prove

THEOREM 5. *If* $K(t)=t^{\lambda_k}$, $0<\lambda_k\to b$ $(k\to\infty)$, $0<b<\infty$ *with* $\lambda_i\neq\lambda_j$ $(i\neq j)$ *and* $D_k>0$ $(k=1,2,\ldots)$, *then* $P_{K,D}$ *is dense in* $C_0[0,1]$ *if and only if*

$$\sum_{k=1}^{\infty}D_k\,|\lambda_k-b|^n=\infty \qquad \text{for all} \quad n=0,1,2,\ldots. \tag{3}$$

Finally, we prove for another type of function K_k

THEOREM 6. *Suppose that the function* ϕ, $\phi(z)=\sum_{\nu=1}^{\infty}b_\nu z^\nu$ *is regular for* $|z|<r$, $0<r\leqslant\infty$ *with*

$$\sum_{\substack{\nu\geqslant 1\\ b_\nu\neq 0}}\nu^{-1}=\infty, \tag{4}$$

and that $0<\lambda_k\to\infty$ $(k\to\infty)$, $\lambda_k^{-1}<r$ $(k=1,2,\ldots)$. *Then for* $K_k(t)=\phi(t\lambda_k^{-1})$ *and* $D_k>0$ $(k=1,2,\ldots)$ *the set* $P_{K,D}$ *is dense in* $C_0[0,1]$ *if and only if*

$$\sum_{k=1}^{\infty}D_k\lambda_k^{-n}=\infty \qquad \text{for all} \quad n=1,2,\ldots. \tag{5}$$

Theorem 6 can be applied, for example, if $K_k(t)=t/(t+\lambda_k)$, $\sin(t\lambda_k^{-1})$, $\exp(t\lambda_k^{-1})-1$ with $\exp(x)=e^x$.
All of our proofs are based upon the following general theorem of functional analysis.

THEOREM 7. *Suppose that* X *is a normed linear space with a norm* $\|x\|$ *for* $x\in X$, *and that* $\tilde{X}\subseteq X$ *is a linear subspace with a seminorm* p *on* $\tilde{X}$. *If*

$G \subseteq X$, then, in order that for every $y \in G$ and $\varepsilon > 0$ there exists an element $x \in \tilde{X}$ satisfying

$$\| y - x \| < \varepsilon \qquad \text{and} \qquad p(x) < \varepsilon,$$

the condition (I_G) is necessary and sufficient, where (I_G) denotes the property that

$$|F(x)| \leqslant Mp(x) \qquad (x \in \tilde{X})$$

for a bounded linear functional F on X always implies $F(y) = 0$ $(y \in G)$, where the constant $M < \infty$ is independent of x.

Theorem 7 is a corollary of a more general result [5, Theorem 10 in the case $Y = X$, $A(x) = x$]. We deduce from Theorem 7 the following main theorem for approximations in normed linear spaces by linear combinations with restricted coefficients.

THEOREM 8. *Let X be a normed linear space, let $x_k \in X$ and $D_k > 0$ $(k = 1, 2, \ldots)$, and let P denote the set of all linear combinations $x = \sum_{k=1}^{N} a_k x_k$ (a_k complex) with the restriction that $|a_k| \leqslant D_k$ $(k = 1, 2, \ldots, N)$. Then P is dense in a linear subset $G \subseteq X$ in the norm of X if and only if the condition (B_G) is satisfied, where (B_G) denotes the property that*

$$\sum_{k=1}^{\infty} D_k \, |F(x_k)| < \infty$$

for a bounded linear functional F on X always implies $F(y) = 0$ $(y \in G)$.

Proof of Theorem 8. The density of P in G is equivalent to the approximation $\| y - x \| < \varepsilon$ with $|a_k| \leqslant \varepsilon D_k$ $(k = 1, 2, \ldots, N)$ for each $y \in G$, $\varepsilon > 0$ with some $x \in P$, since G is linear. This is obvious if y is replaced by $\varepsilon^{-1} y$. Thus we take $\tilde{X}$ in Theorem 7 to be the set of all $x = \sum_{k=1}^{N} a_k x_k$ (without any restriction) and choose the seminorm $p(x) = \max_{1 \leqslant k \leqslant N} |a_k| \, D_k^{-1}$ (which also is a norm on $\tilde{X}$). We obtain in this case the equivalence of the conditions (I_G) of Theorem 7 and (B_G) of Theorem 8. For

$$|F(x)| = \left| \sum_{k=1}^{N} a_k F(x_k) \right| \leqslant M \max_{1 \leqslant k \leqslant N} |a_k| \, D_k^{-1}$$

(a_k complex; $N = 1, 2, \ldots$) implies, in particular, $\sum_{k=1}^{N} D_k \, |F(x_k)| \leqslant M$ for all $N = 1, 2, \ldots$, and so $\sum_{k=1}^{\infty} D_k \, |F(x_k)| < \infty$, if we choose a_k such that $|a_k| = D_k$ and $a_k F(x_k) = D_k \, |F(x_k)|$ $(k = 1, \ldots, N)$. Thus (B_G) implies

(I_G). Conversely, $\sum_{k=1}^{\infty} D_k |F(x_k)| = M < \infty$ implies $|\sum_{k=1}^{N} a_k F(x_k)| \leqslant \sum_{k=1}^{N} D_k |F(x_k)| |a_k| D_k^{-1} \leqslant M \max_{1 \leqslant k \leqslant N} |a_k| D_k^{-1}$ (a_k complex; $N = 1, 2, ...$). Therefore (B_G) is a consequence of (I_G), which proves Theorem 8.

Theorem 2 follows immediately from Theorem 8 in the case $X = C[a, b]$ and $G = C_0[a, b]$, $C[a, b]$ respectively, by the Riesz representation theorem [8, p. 139]. The part of Theorem 2 concerning the density in $C[a, b]$ is obvious, since $\sum_{k=1}^{\infty} D_k |K_k(a)| |\alpha(a+0) - \alpha(a)| < \infty$ implies $\alpha(a) = \alpha(a+0)$ if and only if $\sum_{k=1}^{\infty} D_k |K_k(a)| = \infty$.

The following known theorem, which is used in the proofs of Theorems 3–6, is due to [6].

THEOREM 9. *If* $\lambda_k > 0$, $\lambda_{k+1} - \lambda_k \geqslant c > 0$, *and* $\sum_{k=1}^{\infty} \lambda_k^{-1} = \infty$, *then* $\int_0^1 t^{\lambda_k} d\alpha(t) = \mathcal{O}(q^{\lambda_k})$ $(k \to \infty)$, $\int_0^1 |d\alpha(t)| < \infty$ *implies* $\alpha(t) = 0$ *on* $(q, 1]$ *for each* $q \in (0, 1]$, *where* α *is normalized.*

An alternative proof of Theorem 9 is given in [3, Corollary of Theorem 3].

Proof of Theorem 3. We first assume the existence of a subsequence (k_i) satisfying (1), and verify (B) of Theorem 2 for $K_k(t) = t^{\lambda_k}$, $D_k = A_k^{\lambda_k}$. It follows from $\sum_{k=1}^{\infty} A_k^{\lambda_k} |\int_0^1 t^{\lambda_k} d\alpha(t)| < \infty$ that $\int_0^1 t^{\lambda_{k_i}} d\alpha(t) = \mathcal{O}(A_{k_i}^{-\lambda_{k_i}})$ $(i \to \infty)$. Thus $\alpha(t) = 0$ on $(0, 1]$ by Theorem 9 for a normalized α, and $P_{K,D}$ is dense in $C_0[0, 1]$ by Theorem 2. Conversely, we suppose the density of $P_{K,D}$ in $C_0[0, 1]$. A simple argument shows that there is a subsequence (k_i) with (1) if and only if $\sum_{k \geqslant 1, A_k \geqslant b} \lambda_k^{-1} = \infty$ for all $b > 0$, and we now assume that

$$\sum_{\substack{k \geqslant 1 \\ A_k \geqslant b}} \lambda_k^{-1} < \infty \tag{6}$$

for some $b > 0$. For each $g \in P_{K,D}$ we have

$$g(t) = \sum_{k=1}^{N} a_k t^{\lambda_k} = \sum_{\substack{k \\ A_k < b}} a_k t^{\lambda_k} + \sum_{\substack{k \\ A_k \geqslant b}} a_k t^{\lambda_k} = g_1(t) + g_2(t).$$

Since $|a_k| \leqslant A_k^{\lambda_k}$, $\lambda_{k+1} - \lambda_k \geqslant c > 0$, and so $\lambda_k \geqslant (k-1)c$, we can find a number $d \in (0, 1)$ such that $|g_1(t)| \leqslant \sum_{k=1}^{\infty} b^{\lambda_k} t^{\lambda_k} \leqslant 1$ for all $t \in [0, d]$ and all $g \in P_{K,D}$. By the density of $P_{K,D}$ there is for each $f \in C_0[0, 1]$ and $\varepsilon > 0$ a function $g \in P_{K,D}$, $g = g_1 + g_2$ such that $|2\varepsilon^{-1} f(t) - g_1(t) - g_2(t)| < 1$ $(t \in [0, 1])$. Thus $|2\varepsilon^{-1} f(t) - g_2(t)| < 1 + |g_1(t)| \leqslant 2$, and so $|f(t) - 2^{-1}\varepsilon g_2(t)| < \varepsilon$ for all $t \in [0, d]$. Therefore the set of all g_2 is dense in $C_0[0, d]$, and, referring to the definition of g_2, we have

$\sum_{k \geq 1, A_k \geq b} \lambda_k^{-1} = \infty$ by the theorem of Müntz [8, p. 336], which contradicts (6). This proves Theorem 3.

Proof of Theorem 4. By using a simple substitution we may plainly suppose that $0 < a < b = 1$. We assume first the existence of a subsequence (k_i) with (2), and verify (B) of Theorem 2 for $K_k(t) = t^{\lambda_k}$, $D_k = c_k a^{-\lambda_k}$. Suppose now that

$$\sum_{k=1}^{\infty} c_k a^{-\lambda_k} \left| \int_a^1 t^{\lambda_k} \, d\alpha(t) \right| < \infty, \qquad \int_a^1 |d\alpha(t)| < \infty \tag{7}$$

for a normalized α. For each $\varepsilon > 0$ with $a < 1 - \varepsilon$ we find by (2) an i_0 such that $c_{k_i} > (1-\varepsilon)^{\lambda_{k_i}}$ $(i \geq i_0)$, and, by (7), we have $\int_a^1 t^{\lambda_{k_i}} \, d\alpha(t) = \mathcal{O}((1-\varepsilon)^{-\lambda_{k_i}} a^{\lambda_{k_i}})$ $(i \to \infty)$, which implies $\alpha(t) = 0$ on $[a(1-\varepsilon)^{-1}, 1]$, and so $\alpha(t) = 0$ on $(a, 1]$ by Theorem 9. This proves (B). Conversely, we assume the density of $P_{K,D}$ in $C_0[a, 1]$. The existence of a subsequence (k_i) satisfying (2) is equivalent to the property that $\sum_{k \geq 1, c_k \geq q^{\lambda_k}} \lambda_k^{-1} = \infty$ for all $q \in (0, 1)$. Suppose now that

$$\sum_{\substack{k \geq 1 \\ c_k \geq q^{\lambda_k}}} \lambda_k^{-1} < \infty \tag{8}$$

for some $q \in (0, 1)$. For each $g \in P_{K,D}$ we have

$$g(t) = \sum_{k=1}^{N} a_k t^{\lambda_k} = \sum_{\substack{k \\ c_k < q^{\lambda_k}}} a_k t^{\lambda_k} + \sum_{\substack{k \\ c_k \geq q^{\lambda_k}}} a_k t^{\lambda_k} = g_1(t) + g_2(t). \tag{9}$$

We can choose $d \in (a, 1]$ such that $qa^{-1}d < 1$. Since $|a_k| \leq c_k a^{-\lambda_k}$, and $\lambda_k \geq (k-1)c$, it follows from (9) that

$$|g_1(t)| < \sum_{k=1}^{\infty} q^{\lambda_k} a^{-\lambda_k} d^{\lambda_k} = M < \infty \tag{10}$$

with a constant M for all $t \in [a, d]$ and all $g \in P_{K,D}$. For each $f \in C_0[a, 1]$ and $\varepsilon > 0$ there is a function $g \in P_{K,D}$, $g = g_1 + g_2$ such that $|2\varepsilon^{-1} Mf(t) - g_1(t) - g_2(t)| < M$ $(t \in [a, 1])$, and therefore, by (10), $|2\varepsilon^{-1} Mf(t) - g_2(t)| < 2M$, and so $|f(t) - 2^{-1}\varepsilon M^{-1} g_2(t)| < \varepsilon$ on $[a, d]$. This density of the set of all g_2 in $C_0[a, d]$ implies the divergence of the series in (8) by (9) and by the theorem of Müntz for the interval $[a, d]$, which contradicts (8). Thus, by Theorem 2, condition (B) is equivalent to the existence of a subsequence (k_i) with (2), and therefore the part of Theorem 4 concerning the density in $C[a, b]$ is obvious by Theorem 2. This completes the proof of Theorem 4.

Proof of Theorem 5. First we suppose (3), and prove (B). If

$$\sum_{k=1}^{\infty} D_k \left| \int_0^1 t^{\lambda_k} \, d\alpha(t) \right| < \infty, \qquad \int_0^1 |d\alpha(t)| < \infty \tag{11}$$

for a normalized α, then we set

$$w(s) = \int_0^1 t^s \, d\alpha(t), \qquad s = x + iy.$$

The function w is regular for $x = \mathrm{Re}(s) > 0$, and w has a power series

$$w(s) = \sum_{\nu=0}^{\infty} q_\nu (s-b)^\nu \qquad (|s-b| < b) \tag{12}$$

with coefficients q_ν. Thus $w(\lambda_k) = \int_0^1 t^{\lambda_k} \, d\alpha(t) \to q_0$ $(k \to \infty)$, since $\lambda_k \to b$ $(k \to \infty)$. Therefore it follows from (11) that $\sum_{k=1}^{\infty} D_k \, |q_0| < \infty$, which implies $q_0 = 0$ by (3) in the case $n = 0$. If $q_\nu = 0$ $(\nu = 0, 1, ..., n-1)$ is proved already, then we have $w(\lambda_k) = (\lambda_k - b)^n \, q_n e_k$ with $e_k \to 1$ $(k \to \infty)$ by (12), and therefore $\sum_{k=1}^{\infty} D_k \, |\lambda_k - b|^n \, |q_n| < \infty$ by (11), which implies $q_n = 0$ by (3). Thus $w(s) = 0$ for $\mathrm{Re}(s) > 0$, and we obtain $\alpha(t) = 0$ on $(0, 1]$ by well known identity theorems of the Laplace transform. This proves the density of $P_{K,D}$ in $C_0[0, 1]$ by Theorem 2. We now suppose that $P_{K,D}$ is dense in $C_0[0, 1]$ and prove (3). If

$$\sum_{k=1}^{\infty} D_k \, |\lambda_k - b|^n < \infty \tag{13}$$

for some n, then we choose α so that $w(s) = \int_0^1 t^s \, d\alpha(t) = [1 - e^{-(s-b)}]^n$. Hence $\int_0^1 |d\alpha(t)| < \infty$, and $w(\lambda_k) = (\lambda_k - b)^n \, e_k$ with $e_k \to 1$ $(k \to \infty)$. We obtain $\sum_{k=1}^{\infty} D_k \, |w(\lambda_k)| < \infty$ by (13), where α is not identically 0 on $(0, 1]$. Thus condition (B) is not satisfied, which contradicts the density of $P_{K,D}$. This proves Theorem 5.

Proof of Theorem 6. First, we assume (5). We suppose that

$$\sum_{k=1}^{\infty} D_k \, |H_k| < \infty, \tag{14}$$

with

$$H_k = \int_0^1 \phi(t\lambda_k^{-1}) \, d\alpha(t) = \sum_{i=1}^{\infty} b_{\nu_i} \lambda_k^{-\nu_i} \mu_i, \qquad \mu_i = \int_0^1 t^{\nu_i} \, d\alpha(t), \qquad \int_0^1 |d\alpha(t)| < \infty, \tag{15}$$

where we denote by ν_i with $\nu_{i+1} > \nu_i$ the integers ν for which $b_\nu \neq 0$. To prove the density of $P_{K,D}$ in $C_0[0, 1]$ we have to show, by using Theorem 2, Theorem 9, and (4), that $\mu_i = 0$ $(i = 1, 2, ...)$. Since $\lambda_k \to \infty$, it follows from (15) that $H_k = b_{\nu_1}\mu_1\lambda_k^{-\nu_1}e_k$ with $e_k \to 1$ $(k \to \infty)$. Hence, by (14), $\sum_{k=1}^{\infty} |b_{\nu_1}| |\mu_1| D_k\lambda_k^{-\nu_1} < \infty$, which implies $\mu_1 = 0$ by (5), since $b_{\nu_1} \neq 0$. If $\mu_i = 0$ $(i = 1, ..., n-1)$ is proved already, then, repeating the argument, we obtain $H_k = b_{\nu_n}\mu_n\lambda_k^{-\nu_n}e_k$, $e_k \to 1$ $(k \to \infty)$, and $\sum_{k=1}^{\infty} |b_{\nu_n}| |\mu_n| D_k\lambda_k^{-\nu_n} < \infty$ by (14) and (15), which implies $\mu_n = 0$ by (5), since $b_{\nu_n} \neq 0$. Conversely, suppose that $P_{K,D}$ is dense in $C_0[0, 1]$. To prove (5) we assume that $\sum_{k=1}^{\infty} D_k\lambda_k^{-n} < \infty$ for some n and fix an integer $m > 1$ such that $\nu_m > n$. Since $\lambda_k \to \infty$, we have

$$\sum_{k=1}^{\infty} D_k\lambda_k^{-\nu_m} < \infty. \tag{16}$$

We choose α so that $\int_0^1 t^s\, d\alpha(t) = \prod_{i=1}^{m-1} [1 - e^{-(s-\nu_i)}]$. Thus $\int_0^1 |d\alpha(t)| < \infty$ and $\mu_i = 0$ $(i = 1, ..., m-1)$, where α is not identically 0 on $(0, 1]$. Concerning this α we have by (15)

$$\sum_{k=1}^{\infty} D_k |H_k| \leqslant \sum_{k=1}^{\infty} D_k\lambda_k^{-\nu_m} \sum_{i=m}^{\infty} |b_{\nu_i}| \lambda_k^{\nu_m - \nu_i} |\mu_i|,$$

where (17)

$$\sum_{i=m}^{\infty} |b_{\nu_i}| \lambda_k^{\nu_m - \nu_i} |\mu_i| \leqslant \sum_{i=m}^{\infty} |b_{\nu_i}| (2^{-1}r)^{\nu_i - \nu_m} |\mu_i| \qquad (k \geqslant k_0)$$

for large enough k_0, since $\lambda_k \to \infty$. We have $|\mu_i| \leqslant \int_0^1 |d\alpha(t)| < \infty$. Thus $\sum_{k=1}^{\infty} D_k |H_k| < \infty$ by (16) and (17), and condition (B) of Theorem 2 is not satisfied, in contradiction to the density of $P_{K,D}$ in $C_0[0, 1]$. This completes the proof of Theorem 6.

Analogous theorems can be proved by Theorem 7 or Theorem 8 for the approximation in $L^p[a, b]$-spaces or normed linear spaces, provided that the linear functionals have an explicit representation, where we might choose restrictions on the coefficients by various seminorms.

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